# Math 3033 Tutorial 7

**Theorem** (Weyl). *Let $\{x_n\}$ be a sequence of real numbers in $[0,1)$. Then the following are equivalent:*

*(a) $\{x_n\}$ is equidistributed;*

$\boxed{(b)}$ *for any $k \in \mathbb{Z} \setminus \{0\}$,*

Useful criterion

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} \to 0;$$

*(c) for any 1-periodic continuous function $f$,*

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) \to \int_0^1 f(y)dy.$$

All conditions above can be rewritten as

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) \longrightarrow \int_0^1 f \quad \text{as} \quad N \to \infty$$

(a) $\chi_I$ : $I \subseteq [0,1)$

(b) All trigonometric polynomials

(c) 1-periodic Cont. fcn.

Pf : $\chi_I \longrightarrow$ Step fcns $\longrightarrow$ Riemann integrable $\longrightarrow$ cont fcns $\longrightarrow \chi_I$

cont fcns $\Updownarrow$ Fejér's kernel

Trigo poly

( uniform approx. of periodic cont. fcns by trigo polys )

Let $a \neq 0$ and $0 < \sigma < 1$. Show the the sequence $\langle an^\sigma \rangle$ is equidistributed in $[0,1)$. Here $\langle x \rangle$ denotes the fractional part of $x$.

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$\langle x \rangle$ = fractional part of $x$

$= x - \lfloor x \rfloor$

$\lfloor x \rfloor$ := integer part of $x$

:= the greatest integer that is less than $x$

$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$

$\iff \quad 0 \leq \langle x \rangle < 1$

**Proof**: First fix $k \in \mathbb{Z} \setminus \{0\}$,

$$\frac{1}{N} \sum_{n=1}^{N} e^{i 2\pi k \langle a n^\sigma \rangle} \to 0 \text{ as } N \to \infty$$

$$\| $$

$$\frac{1}{N} \sum_{n=1}^{N} e^{i 2\pi \underbrace{k a}_{} n^\sigma} \approx \frac{1}{N} \sum_{n=1}^{N} \int_n^{n+1} e^{i 2\pi b x^\sigma} dx$$

$(b = ka)$

Showing that $\sum_{n=1}^{N} e^{i2\pi b n^{\sigma}}$ is comparable to $\sum_{n=1}^{N}\int_{n}^{n+1} e^{i2\pi b x^{\sigma}}dx$:

$$\left| \sum_{n=1}^{N} e^{i2\pi b n^{\sigma}} - \sum_{n=1}^{N}\int_{n}^{n+1} e^{i2\pi b x^{\sigma}}dx \right|$$

$$= \left| \sum_{n=1}^{N}\int_{n}^{n+1} e^{i2\pi b n^{\sigma}} - e^{i2\pi b x^{\sigma}} dx \right|$$

$$\leq \sum_{n=1}^{N}\int_{n}^{n+1} \left| e^{i2\pi b n^{\sigma}} - e^{i2\pi b x^{\sigma}} \right| dx$$

$$= \sum_{n=1}^{N}\int_{n}^{n+1} \left| 1 - e^{i2\pi b (x^{\sigma}-n^{\sigma})} \right| dx$$

$$= \sum_{n=1}^{N}\int_{n}^{n+1} \left| e^{-i\pi b (x^{\sigma}-n^{\sigma})} - e^{i\pi b (x^{\sigma}-n^{\sigma})} \right| dx$$

$$= \sum_{n=1}^{N}\int_{n}^{n+1} 2\left| \sin \pi b (x^{\sigma}-n^{\sigma}) \right| dx$$

$$\leq \sum_{n=1}^{N}\int_{n}^{n+1} 2\pi b (x^{\sigma}-n^{\sigma})\, dx \qquad (|\sin x| \leq |x|)$$

$$\leq 2\pi b \sum_{n=1}^{N} (n+1)^{\sigma} - n^{\sigma} = 2\pi b\left[(N+1)^{\sigma} - 1\right]$$

Since $0<\sigma<1$,

$$\left| \frac{1}{N}\sum_{n=1}^{N} e^{i2\pi b n^{\sigma}} - \sum_{n=1}^{N}\int_{n}^{n+1} e^{i2\pi b x^{\sigma}}dx \right| \to 0 \quad \text{as } N \to \infty$$

$$\sum_{n=1}^{N} \int_n^{n+1} e^{2\pi i b x^\sigma} \, dx$$

$$= \int_1^{N+1} e^{2\pi i b x^\sigma} \, dx$$

$$y^{\frac{1}{\sigma}} = x$$

$$\Updownarrow$$

$$(\, y = x^\sigma \,)$$

$$= \int_1^{(N+1)^\sigma} e^{2\pi i b y} \, \frac{dy}{\sigma y^{1-\frac{1}{\sigma}}}$$

$$dy = \sigma x^{\sigma-1} \, dx$$

$$dx = \frac{dy}{\sigma x^{\sigma-1}}$$

$$= \frac{dy}{\sigma y^{1-\frac{1}{\sigma}}}$$

$$= \frac{1}{\sigma} \int_1^{(N+1)^\sigma} e^{2\pi i b y} \, y^{\frac{1}{\sigma}-1} \, dy$$

$$= \frac{1}{\sigma} \int_1^{(N+1)^\sigma} \frac{1}{2\pi i b} \, y^{\frac{1}{\sigma}-1} \, d\left(e^{2\pi i b y}\right)$$

$$= \frac{1}{2\pi i b \sigma} \left\{ \left[ y^{\frac{1}{\sigma}-1} e^{2\pi i b y} \right]_{y=1}^{(N+1)^\sigma} - \int_1^{(N+1)^\sigma} \left(\frac{1}{\sigma}-1\right) y^{\frac{1}{\sigma}-2} e^{2\pi i b y} \, dy \right\}$$

$$\left| \left[ y^{\frac{1}{\sigma}-1} e^{2\pi i b y} \right]_{y=1}^{(N+1)^\sigma} \right| = \left| (N+1)^{1-\sigma} e^{2\pi i b (N+1)^\sigma} - e^{2\pi i b} \right|$$

$$\leq (N+1)^{1-\sigma} + 1$$

$$\left| \int_1^{(N+1)^\sigma} \left(\frac{1}{\sigma}-1\right) y^{\frac{1}{\sigma}-2} e^{2\pi i b y} \, dy \right|$$

$$\leq \int_1^{(N+1)^\sigma} \left(\frac{1}{\sigma}-1\right) y^{\frac{1}{\sigma}-2} \cdot 1 \, dy$$

$$= \left(\frac{1}{\sigma}-1\right) \left[ y^{\frac{1}{\sigma}-1} \right]_{y=1}^{(N+1)^\sigma}$$

$$= \left(\frac{1}{\sigma}-1\right) \cdot \left( (N+1)^{1-\sigma} - 1 \right)$$

$$\therefore \quad \frac{1}{N} \int_1^{N+1} e^{2\pi i b x^\sigma} \, dx \to 0 \quad \text{as } N \to \infty$$

Prove that $\langle a \log n \rangle$ is not equidistributed for any $a$.

Ch. 4 Ex 9 : For $k \in \mathbb{Z} \setminus \{0\}$

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \underbrace{ka}\log n} \not\to 0$$

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i b \log n} \not\to 0$$

$$\overset{??}{\frac{1}{N} \sum_{n=1}^{N} \int_{n}^{n+1} e^{2\pi i b \log x}\, dx}$$

① Really comparable

② $$\left| \frac{1}{N} \int_{1}^{N+1} e^{2\pi i b \log x}\, dx \right| \not\to 0$$

as $N \to \infty$